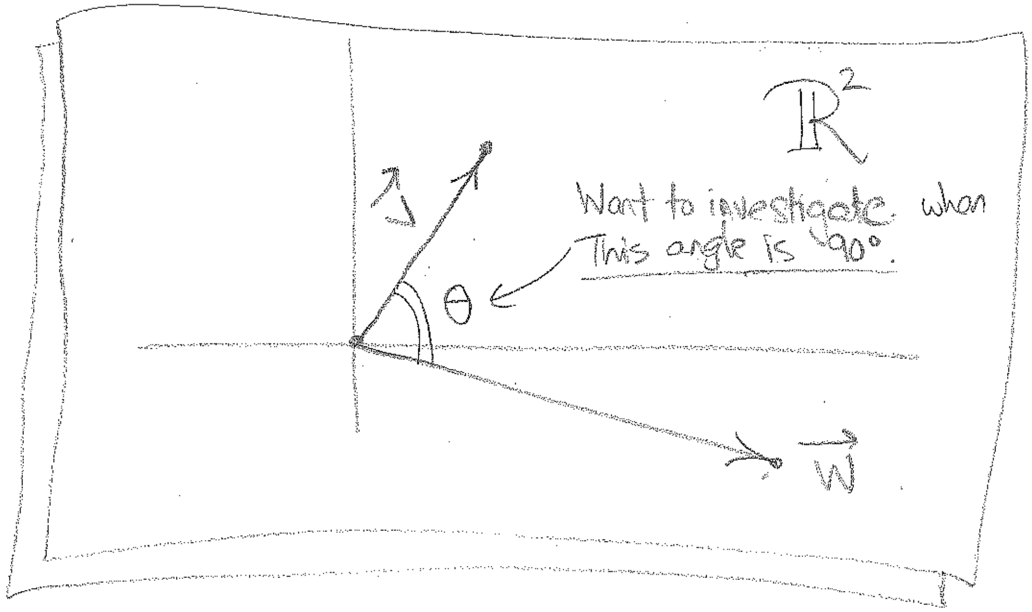
MATH 312

Lecture 12: ANGLES | ORTHOGONALITY

ANGLES & LINEAR ALGEBRA:

Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$.

eg:

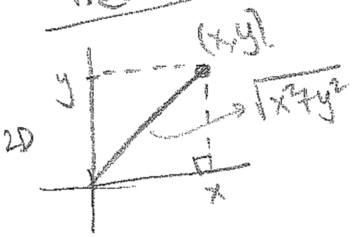


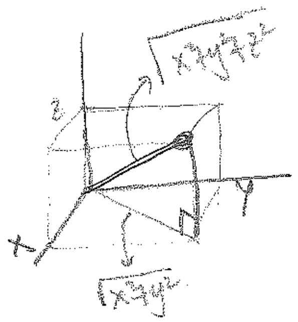
First, we can measure lengths by the usual formula:

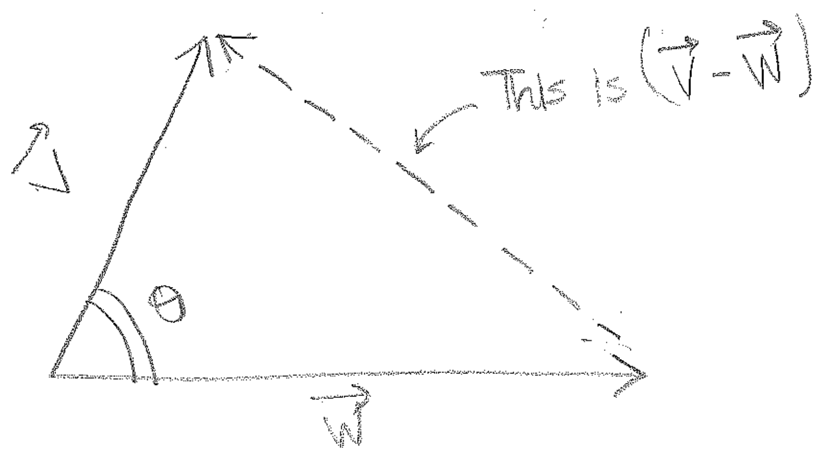
If $\vec{v} = (v_1, \dots, v_n)$, then its length is:

$$\|\vec{v}\| = \left[v_1^2 + v_2^2 + \dots + v_n^2 \right]^{1/2} = \left(\vec{v}^T \cdot \vec{v} \right)^2$$

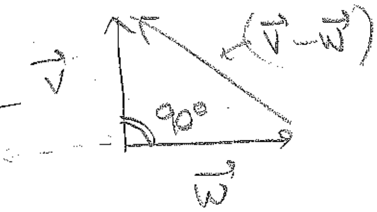
By Pythagoras' Theorem:







If the angle between $\vec{v}$ and $\vec{w}$ is $90°$, then Pythagoras' Theorem applies to the triangle with sides $\vec{v}$, $\vec{w}$ and $\vec{v}-\vec{w}$:



$$\|v\|^2 + \|w\|^2 = \|v-w\|^2$$

Or, $(v_1^2 + \cdots + v_n^2) + (w_1^2 + \cdots + w_n^2) = (v_1-w_1)^2 + (v_2-w_2)^2 + \cdots + (v_n-w_n)^2$

The Right Side equals the left side MINUS

$2(v_1w_1 + \cdots + v_nw_n)$ ⟵ This must be zero if $\theta = 90°$

BUT: The quantity $(v_1w_1 + \cdots + v_nw_n)$ is precisely $\vec{v}^T\vec{w}$ or $\vec{w}^T\vec{v}$! ⟶ This is the "INNER PRODUCT" or "DOT PRODUCT" of $\vec{v}$ and $\vec{w}$.

(they are the same)

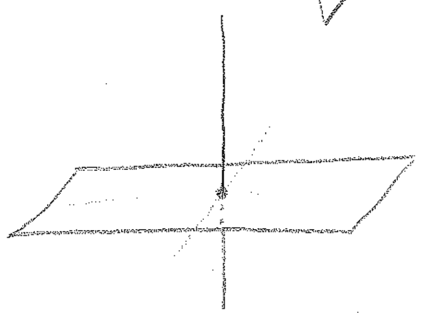
$$[v_1 \dots v_n]\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1w_1 + \cdots + v_nw_n$$

So, $\vec{v}$ and $\vec{w}$ are "perpendicular" or "orthogonal" or "at right angles" if and only if $\vec{v}^T\cdot\vec{w}$ (or $\vec{w}^T\cdot\vec{v}$ equal zero).

And Two SUBSPACES $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{R}^n$ are perpendicular or ... etc. if every vector v in $\mathbb{V}$ is perpendicular to every vector w in $\mathbb{W}$. (Test bases!)

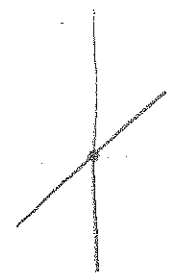
WARNING By this definition the $\vec{0}$ vector AND the $\vec{0}$ subspace are perpendicular to EVERYTHING

Eg



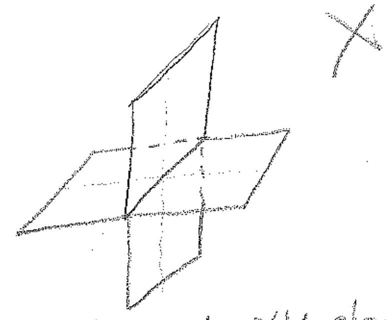
z - axis & XY plane

A



z Axis & X Axis

B



XZ plane & XY plane

(Not perpendicular thanks to X Axis)

C

AMAZING FACT: If $\vec{V}_1, \ldots, \vec{V}_k$ are NON-ZERO vectors in $\mathbb{R}^n$ so that ANY PAIR is orthogonal, then the $\vec{V}$'s are linearly independent!

"Proof" If $\vec{W} = c_1\vec{V}_1 + \cdots + c_k\vec{V}_k = \vec{0}$

then $\vec{W}^T = c_1\vec{V}_1^T + \cdots + c_k\vec{V}_k^T = \vec{0}^T$

So, $0 = \vec{W}^T\vec{W} = c_1^2\|\vec{V}_1\|^2 + \cdots + c_k^2\|V_k\|^2$

$$\underbrace{+\,2c_1c_2\,\vec{V}_1^T\vec{V}_2 + 2c_1c_3\,\vec{V}_1^T\vec{V}_3 + \cdots + 2c_{k-1}c_k\,\vec{V}_{k-1}^T\vec{V}_k}_{\text{These are ALL zero by orthogonality}}$$

So, $0 = c_1^2\|\vec{V}_1\|^2 + \cdots + c_k^2\|V_k\|^2$

Since EVERYTHING on the right side is ≥ 0, and since each $\|V_\circ\|^2$ is nonzero, the $c_\circ^2$'s MUST be zero!! So, the $\vec{V}$'s are INDEPENDENT.

"ORTHOGONAL COMPLEMENT"

Def Let V be a subspace of $\mathbb{R}^n$. Then the collection of ALL VECTORS ORTHOGONAL TO V is also a subspace of $\mathbb{R}^n$, written $V^\perp$ (V-perp). (Also, $(V^\perp)^\perp = V$!) : (intersect at $\vec{0}$)

Eg. In examples A, B and C: only A shows an example of V = z-axis and $V^\perp$ = XY plane, NOT B!

BACK TO THE FTLA:

If A is an $m \times n$ matrix, then:

- $N(A)$ and $C(A^T)$ are subspaces of $\mathbb{R}^n$
- $N(A^T)$ and $C(A)$ are subspaces of $\mathbb{R}^m$
- $\dim N(A) + \dim C(A^T) = n$
- $\dim C(A) + \dim N(A^T) = m$

KNOW THIS

But there's MORE:

$$\boxed{\begin{aligned} N(A) &= C(A^T)^\perp \\ \text{and} \quad C(A) &= N(A^T)^\perp \end{aligned}}$$

<u>Pf</u> $N(A) = \{\vec{x} \text{ in } \mathbb{R}^n \text{ so } A\vec{x} = \vec{0}\}$

$C(A^T) = \{\text{span of rows of } A\}$

$A\vec{x} = 0$, so:

$$\begin{bmatrix} \text{— row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row m —} \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

So, $(\text{row } 1)^T \cdot \vec{x} = 0$, and so on for every row. This means:

$\left[\text{Every row of } A \text{ is orthogonal to every } \vec{x} \text{ in the null space of } A.\right]$

So, $N(A)$ is orthogonal to $C(A^T)$!

<u>BUT</u> to get $N(A) = C(A^T)^\perp$, we also need to count dimensions!